Application of reciprocal Backlund transformations to a class of nonlinear boundary value problems

1983 J. Phys. A: Math. Gen. 16 L493

(http://iopscience.iop.org/0305-4470/16/14/001)

J. Phys. A: Math. Gen. 16 (1983) L493–L495. Printed in Great Britain

LETTER TO THE EDITOR

Application of reciprocal Bäcklund transformations to a class of nonlinear boundary value problems

C Rogers

Department of Applied Mathematics, University of Waterloo, Waterloo, Canada

Received 8 June 1983

Abstract. It is shown that reciprocal Bäcklund transformations may be used to reduce an important class of nonlinear boundary value problems for higher-order nonlinear diffusion equations to linear canonical boundary value problems.

In a recent paper (Kingston and Rogers 1982) reciprocal Bäcklund transformations were established for a broad class of conservation laws. The result was used to generate auto-Bäcklund transformations for reciprocally associated nonlinear evolution equations and to construct permutability diagrams for the generation of solutions. Here, it is shown that the reciprocal Bäcklund transformations may also be used to reduce an important class of nonlinear boundary value problems to linear canonical forms. The latter may be solved by standard integral transform methods.

In what follows, we shall make use of the following consequence of a more general reciprocal property established by Kingston and Rogers (1982).

Theorem. The conservation law

$$\partial u/\partial t + (\partial/\partial x)\mathscr{C}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n) = 0$$

is transformed under the Bäcklund transformation

$$\mathrm{d}x' = u\,\mathrm{d}x - \mathscr{C}(u, \partial u/\partial x, \ldots, \partial^n u/\partial x^n)\mathrm{d}t, \qquad t' = t, \qquad u' = 1/u, \quad \text{(R)}$$

to the reciprocally associated conservation law

$$\partial u'/\partial t' + \partial\mathscr{C}'/\partial x' = 0,$$

where

$$\mathscr{C}' = -u'\mathscr{C}(\mathbb{D}'^{(0)}(1/u'), \mathbb{D}'^{(1)}(1/u'), \ldots, \mathbb{D}'^{(n)}(1/u'))$$

and $\mathbb{D}' \equiv (1/u')\partial/\partial x'$.

The intrinsic reciprocity of the above result derives from the fact $R^2 = I$. Extensions may be made to systems of conservation laws (Kingston and Rogers 1983). Such results extend well known reciprocal invariance properties in gas dynamics (Rogers 1968, 1969, Rogers and Shadwick 1982).

It is shown here that the above reciprocal property may be used to reduce to linear canonical form the following class of nonlinear boundary value problems

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left[u\sum_{i=1}^{n}\alpha_i\mathbb{D}^i\left(\frac{1}{u}\right)\right]=0, \qquad 0<x<L, \qquad t>0, \tag{1}$$

$$u\sum_{i=1}^{n}\alpha_i\mathbb{D}^i\left(\frac{1}{u}\right)=\Phi(t) \qquad \text{at } x=0, \qquad t>0, \tag{2}$$

$$u\sum_{i=1}^{n}\alpha_i\mathbb{D}^i\left(\frac{1}{u}\right)=-\Phi(t) \qquad \text{at } x=L, \qquad t>0, \tag{3}$$

$$u=\bar{u} \qquad \text{at } t=0, \qquad 0\leqslant x\leqslant L, \tag{4}$$

where $\mathbb{D}\equiv(1/u)\partial/\partial x$. The nonlinear boundary conditions (2), (3) correspond to prescribed flux at the boundaries $x=0$ and $x=L$ respectively and $\bar{u}>0$ is a constant.

Application of the reciprocal Bäcklund transformation

$$\mathrm{d}x'=u\,\mathrm{d}x+u\sum_{i=1}^{n}\alpha_i\mathbb{D}^i\left(\frac{1}{u}\right)\mathrm{d}t, \qquad t'=t, \tag{5}$$

$$u'=1/u, \tag{6}$$

to the nonlinear evolution equation (1) produces the *linear* equation

$$\frac{\partial u'}{\partial t'}+\frac{\partial}{\partial x'}\left(\sum_{i=1}^{n}\alpha_i\frac{\partial^i u'}{\partial x'^i}\right)=0, \qquad t'>0. \tag{7}$$

In particular, if $n=2$ we obtain the classical $1+1$ heat equation while if $n=3$ the linearised Korteweg–deVries equation is retrieved.

Now, (5) shows that

$$\partial x'/\partial x=u,$$

$$\frac{\partial x'}{\partial t}=u\sum_{i=1}^{n}\alpha_i\mathbb{D}^i\left(\frac{1}{u}\right)=\int_0^x\frac{\partial}{\partial x}\left(u\sum_{i=1}^{n}\alpha_i\mathbb{D}^i\left(\frac{1}{u}\right)\right)\mathrm{d}x+\Phi(t)=\int_0^x\frac{\partial u}{\partial t}\,\mathrm{d}x+\Phi(t)$$

whence

$$x'\equiv x'(x,t)=\int_0^x u\,\mathrm{d}x+\Psi(t)+K, \tag{8}$$

where $\mathrm{d}\Phi/\mathrm{d}t\equiv\Psi$ and K is a constant. If we take $x'(0,0)=0$ then $K=-\Psi(0)$ and we obtain

$$x'=\int_0^x u\,\mathrm{d}x+\Psi(t)-\Psi(0). \tag{9}$$

Thus, the nonlinear boundary condition (2) reduces to the *linear* boundary condition

$$\sum_{i=1}^{n}\alpha_i\frac{\partial^i u'}{\partial x'^i}=u'\Phi(t') \qquad \text{at } x'=\Psi(t')-\Psi(0), \qquad t'>0, \tag{10}$$

under the Bäcklund transformation (5)–(6). Further, the flux conditions (2) and (3) at $x=0$ and $x=L$ respectively show that

$$\left[u \sum_{i=1}^{n} \alpha_i \mathbb{D}^i\left(\frac{1}{u}\right)\right]_0^L = \int_0^L \frac{\partial u}{\partial t}\,\mathrm{d}x = \frac{\partial}{\partial t}\int_0^L u\,\mathrm{d}x = -2\Phi(t) \tag{11}$$

whence

$$\int_0^L u\,\mathrm{d}x = -2\Psi(t) + M \tag{12}$$

where the initial condition (4) shows that

$$M = \bar{u}L + 2\Psi(0). \tag{13}$$

Accordingly, the nonlinear boundary condition (3) reduces under (5)–(6) to the *linear* boundary condition

$$\sum_{i=1}^{n} \alpha_i \frac{\partial^i u'}{\partial x'^i} = -u'\Phi(t') \qquad \text{at } x' = \Psi(0) - \Psi(t') + \bar{u}L, \qquad t' > 0. \tag{14}$$

To summarise, it has been shown that under the reciprocal Bäcklund transformation (5)–(6) the nonlinear boundary value problem (1)–(4) reduces to the *linear* boundary value problem

$$\begin{aligned}
&\frac{\partial u'}{\partial t'} + \sum_{i=1}^{n} \alpha_i \frac{\partial^{i+1} u'}{\partial x'^{i+1}} = 0, && t' > 0, \\
&\sum_{i=1}^{n} \alpha_i \frac{\partial^i u'}{\partial x'^i} = u'\Phi(t') \qquad \text{at } x' = \Psi(t') - \Psi(0), && t' > 0, \\
&\sum_{i=1}^{n} \alpha_i \frac{\partial^i u'}{\partial x'^i} = -u'\Phi(t') \qquad \text{at } x' = \Psi(0) - \Psi(t') + \bar{u}L, && \\
&u' = 1/u \qquad \text{at } t' = 0. &&
\end{aligned} \tag{15}$$

Analogous results may be obtained for nonlinear boundary value problems with prescribed flux on the boundary of a half-space $x > 0$. Reciprocal Bäcklund transformations have recently been employed by Rogers *et al* (1983) to reduce a nonlinear boundary value problem in two-phase flow in a reservoir to a linear canonical boundary value problem. The latter was then solved by standard integral transform techniques.

The author wishes to acknowledge with gratitude his support under Natural Sciences and Engineering Research Council of Canada Grant No A0879.

References

Kingston J G and Rogers C 1982 *Phys. Lett.* **92**A 261
—— 1983 *Quart. Appl. Math.* to be published
Rogers C 1968 *Z. Angew.* (*Math.*) *Phys.* **19** 58
—— 1969 *Z. Angew.* (*Math.*) *Phys.* **20** 370
Rogers C and Shadwick W F 1982 *Bäcklund Transformations and Their Applications* (New York: Academic)
Rogers C, Stallybrass M P and Clements D L 1983 *Nonlinear Analysis, Theory, Methods and Applications* to be published